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An upper bound for the k-barycentric Davenport constant of groups of prime order

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ABSTRACT

Let G be a finite abelian group and let $k \geqslant 2$ be an integer. A sequence of k elements $a_1, a_2, \ldots, a_k$ in G is called a k-barycentric sequence if there exists $j \in \{1, 2, \ldots, k\}$ such that $\sum_{i=1}^{k} a_i = ka_j$. The k-barycentric Davenport constant $\mathrm{BD}(k, G)$ is defined to be the smallest number s such that every sequence in G of length s contains a k-barycentric subsequence. In this paper, we prove that if $p \geqslant 5$ is a prime, then $\mathrm{BD}(k, \mathbb{Z}_p) \leqslant p + k - \left\lfloor \frac{p-2}{k} \right\rfloor - 2$ for $3 \leqslant k \leqslant p - 1$, which improves a result of Delorme et al.

1. Introduction

Let G be a finite abelian group and let $k \geqslant 2$ be an integer. A sequence of k elements $a_1, a_2, \ldots, a_k$ in G is called a *k-barycentric sequence* if there exists $j \in \{1, 2, \ldots, k\}$ such that $\sum_{i=1}^{k} a_i = ka_j$. The *$k$-barycentric Davenport constant* $\mathrm{BD}(k, G)$ is defined to be the smallest number s such that every sequence in G of length s contains a k-barycentric subsequence.

The notion of a barycentric sequence was introduced by Delorme et al. in [5] and was investigated in [4,10,11]; a survey on this topic can be found in [16]. Notice that $a_1, a_2, \ldots, a_k$ is a k-barycentric sequence if and only if there exists $j \in \{1, 2, \ldots, k\}$ such that $a_1 + \cdots + a_{j-1} + (1-k)a_j + a_{j+1} + \cdots + a_k = 0$. Therefore a barycentric sequence is a particular case of zero-sum weighted sequences which were investigated by Hamidoune [13,14], Gao [9], and Grynkiewicz [12]. A comprehensive list of references on zero-sum problems can be found in the surveys [1,8].

The Erdős–Ginzburg–Ziv theorem, which is a starting point of zero-sum problems, now can be restated as follows.

Theorem 1.1 ([7]). *Let $\mathbb{Z}_n$ be the additive group of residue classes modulo n. Then* $\mathrm{BD}(n, \mathbb{Z}_n) = 2n - 1$.

We recall that the *Davenport constant* $\mathrm{D}(G)$ of a finite abelian group G is the smallest number s such that every sequence in G of length s contains a subsequence with zero-sum. The following result of Hamidoune is a generalization of Theorem 1.1.

Theorem 1.2 ([14]). *If G is a finite abelian group, then $\mathrm{BD}(k, G) \leqslant |G| + k - 1$ for every $k \geqslant 2$. Moreover, if $k \geqslant |G|$, then $\mathrm{BD}(k, G) \leqslant \mathrm{D}(G) + k - 1$.*

It is trivial to see that $\mathrm{BD}(2, G) = |G| + 1$ for every finite abelian group G. In the case of the cyclic group $G = \mathbb{Z}_p$ of prime order p, the following result of Delorme et al. is an improvement of Theorem 1.2 for $3 \leqslant k \leqslant p - 1$.

Theorem 1.3 ([4]). *If $p \geqslant 5$ is a prime, then*

(i) $\mathrm{BD}(3, \mathbb{Z}_p) \leqslant 2\lceil p/3 \rceil + 1$,
(ii) $\mathrm{BD}(k, \mathbb{Z}_p) \leqslant p + k - 2$ *for* $4 \leqslant k \leqslant p - 1$,
(iii) $\mathrm{BD}(p - 1, \mathbb{Z}_p) = 2p - 3$.

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The main result of this paper is to prove that if $p \geqslant 5$ is a prime, then

$$\mathrm{BD}(k, \mathbb{Z}_p) \leqslant p + k - \left\lfloor \frac{p-2}{k} \right\rfloor - 2$$

for $3 \leqslant k \leqslant p-1$, which gives an improvement of Theorem 1.3. The paper is organized as follows. Section 2 presents some preliminaries. Section 3 contains our main result and some remarks.

From now on, let p denote a prime. We consider two sequences in $\mathbb{Z}_p$ to be identical if they only differ by the order of their elements and, for convenience, we will use the notation $[a_1]^{\alpha_1}[a_2]^{\alpha_2}\ldots[a_t]^{\alpha_t}$ to denote a sequence in $\mathbb{Z}_p$ where each element a_i appears α_i times.

Throughout this paper, we will denote by $|S|$ the length, by $d(S)$ the number of distinct elements, and by $h(S)$ the maximum multiplicity of an element from a sequence S.

2. Preliminaries

In this section we introduce the tools used to prove the main result of the paper. We first recall the Cauchy–Davenport Theorem and Vosper's Theorem on sumsets.

Theorem 2.1 ([2,3]). *Let $A_1, A_2, \ldots, A_k$, where $k \geqslant 1$, be non-empty subsets of $\mathbb{Z}_p$. Set*

$$A_1 + A_2 + \cdots + A_k = \{a_1 + a_2 + \cdots + a_k \mid a_i \in A_i \text{ for } i = 1, 2, \ldots, k\}.$$

Then $|A_1 + A_2 + \cdots + A_k| \geqslant \min(p, |A_1| + |A_2| + \cdots + |A_k| - (k-1))$.

Theorem 2.2 ([17]). *Let A, B be two subsets of $\mathbb{Z}_p$ with $\min(|A|, |B|) \geqslant 2$. If A and B are not arithmetic progressions with the same common difference, then*

$$|A + B| \geqslant \min(p-1, |A| + |B|).$$

Next we recall the Dias da Silva–Hamidoune Theorem on restricted sumsets, conjectured by Erdő–Heilbronn.

Theorem 2.3 ([6]). *Let A be a subset of $\mathbb{Z}_p$ with $|A| \geqslant 2$. Set*

$$2^\wedge A = \{a + b \mid a \in A, b \in A, a \neq b\}.$$

Then $|2^\wedge A| \geqslant \min(p, 2|A| - 3)$.

Let S be a sequence of elements from a set X. A *k-setpartition* of S is a factorization $S = A_1A_2\cdots A_k$ with $h(A_i) = 1$ for all $i = 1, 2, \ldots, k$. We consider each subsequence A_i to be a non-empty subset, and denote by $A_1, A_2, \ldots, A_k$ the k-setpartition of S. The following simple fact will be frequently used; see for instance [4].

Lemma 2.4. *Let S be a sequence of elements from a set X. If k is an integer with $h(S) \leqslant k \leqslant |S|$, then there exists a k-setpartition $A_1, A_2, \ldots, A_k$ of S such that $|A_i| = \lceil |S|/k \rceil$ or $|A_i| = \lfloor |S|/k \rfloor$ for $i = 1, 2, \ldots, k$.*

We also need the following lemma.

Lemma 2.5. *Let B be a subset of $\mathbb{Z}_p$, where $p \geqslant 5$, with $2 \leqslant |B| \leqslant p-2$. If there are two ways to arrange the elements of B into arithmetic progressions with common differences d_1 and d_2, where $1 \leqslant d_1 \leqslant p-1$ and $1 \leqslant d_2 \leqslant p-1$, then either $d_1 = d_2$ or $d_1 + d_2 = p$.*

Proof. Without loss of generality, we may assume $B = \{0, 1, \ldots, t-1\}$, where $t = |B|$. Suppose, to the contrary, that there is an arrangement of elements of B into an arithmetic progression with common difference d, where $2 \leqslant d \leqslant p-2$. Notice that $|(B + d) \setminus B| \leqslant 1$. We consider two cases for d.

Case 1: $2 \leqslant d \leqslant t$. It is clear that $t \notin B$ and $t + 1 \notin B$ since $t < t + 1 \leqslant p - 1$, and that $t \in B + d$ and $t + 1 \in B + d$ since $0 \leqslant t - d < t - d + 1 \leqslant t - 1$. It follows that $|(B + d) \setminus B| \geqslant 2$, a contradiction.

Case 2: $t \leqslant d \leqslant p - 2$. It is clear that $d \notin B$ and $d + 1 \notin B$ since $t \leqslant d < d + 1 \leqslant p - 1$, and that $d \in B + d$ and $d + 1 \in B + d$ since $0 < 1 \leqslant t - 1$. It follows that $|(B + d) \setminus B| \geqslant 2$, a contradiction.

Thus either $d = 1$ or $d = p - 1$, and the lemma follows. □

3. An upper bound for $\mathrm{BD}(k, \mathbb{Z}_p)$

The main result of the paper is the following theorem whose proof will be given at the end of this section.

Theorem 3.1. *Let $p \geqslant 5$ be a prime and let $3 \leqslant k \leqslant p - 1$ be an integer. Then*

$$\mathrm{BD}(k, \mathbb{Z}_p) \leqslant p + k - \left\lfloor \frac{p-2}{k} \right\rfloor - 2.$$

As an easy consequence of Theorem 3.1, we have the following result, where (ii) holds since the sequence $[0]^{p-3}[1]^{p-3}$ does not contain any $(p-2)$-barycentric subsequence.

Corollary 3.2. *If $p \geqslant 5$ is a prime, then*

(i) $\mathrm{BD}(k, \mathbb{Z}_p) \leqslant p + k - 3$ *for* $3 \leqslant k \leqslant p - 2$,
(ii) $\mathrm{BD}(p - 2, \mathbb{Z}_p) = 2p - 5$.

Remark 3.3. (i) Theorems 1.1 and 1.3(i)(ii) show that $k = p$ is the unique value of k, where $3 \leqslant k \leqslant p$, for which the equality $\mathrm{BD}(k, \mathbb{Z}_p) = p + k - 1$ holds.

(ii) Theorem 1.3(iii) and Corollary 3.2(i) show that $k = p - 1$ is the unique value of k, where $3 \leqslant k \leqslant p - 1$, for which the equality $\mathrm{BD}(k, \mathbb{Z}_p) = p + k - 2$ holds, answering a question raised by Delorme et al. in [4].

(iii) Suggested from Corollary 3.2, we may ask if $k = p - 2$ is the unique value of k, where $3 \leqslant k \leqslant p - 2$, for which the equality $\mathrm{BD}(k, \mathbb{Z}_p) = p + k - 3$ holds. It can be seen that the answer is affirmative for $p = 5$ since $\mathrm{BD}(3, \mathbb{Z}_5) = 5$ as shown in [4]; the answer is negative for $p = 7$ since $\mathrm{BD}(3, \mathbb{Z}_7) = 7$ and $\mathrm{BD}(4, \mathbb{Z}_7) = 8$ as shown in [4], and $\mathrm{BD}(5, \mathbb{Z}_7) = 9$ by Corollary 3.2(ii). We believe that the answer is affirmative for sufficiently large p.

Remark 3.4. It is easy to check that the upper bounds for $\mathrm{BD}(3, \mathbb{Z}_p)$ in Theorems 1.3(i) and 3.1 are the same. As shown in [4], equality occurs in Theorem 1.3(i) for $p \in \{5, 7, 11\}$; however, $\mathrm{BD}(3, \mathbb{Z}_{13}) = 9 < 2\lceil 13/3 \rceil + 1 = 11$ (see also [4]).

We will show that, for sufficiently large p, the upper bound for $\mathrm{BD}(3, \mathbb{Z}_p)$ can be considerably improved. Let $\beta(\mathbb{Z}_p)$ denote the maximal cardinality of a subset $A \subseteq \mathbb{Z}_p$ which does not contain a 3-term arithmetic progression. Then by the pigeon hole principle, $\mathrm{BD}(3, \mathbb{Z}_p) = 2\beta(\mathbb{Z}_p) + 1$. Using a result of Heath-Brown [15], we obtain

$$\mathrm{BD}(3, \mathbb{Z}_p) = O\left(p/(\log p)^{\alpha}\right)$$

for some fixed $\alpha > 0$.

Proof of Theorem 3.1. Set $t = \lfloor (p-2)/k \rfloor + 2$. Let S be a sequence in $\mathbb{Z}_p$ with $|S| = p + k - t$. We will prove that S contains a k-barycentric subsequence. If $h(S) \geqslant k$, then it is clear that S contains a k-barycentric subsequence. So we may assume $h(S) \leqslant k - 1$. A simple computation shows that

$$\begin{aligned} |S| - (k-1)(t-1) &= (p + k - t) - (k-1)(t-1) = (p-1) - k(t-2) \\ &= (p-1) - k\left\lfloor \frac{p-2}{k} \right\rfloor \geqslant (p-1) - (p-2) > 0, \end{aligned}$$

which implies $|S| > (k-1)(t-1)$. Since $h(S) \leqslant k - 1$, it follows that $d(S) \geqslant t$. Suppose that

$$S = [u_1]^{n_1}[u_2]^{n_2} \cdots [u_t]^{n_t} \cdots [u_r]^{n_r},$$

where $r = d(S)$, the elements u_i, for $i = 1, 2, \ldots, r$, are pairwise distinct, and $k - 1 \geqslant n_1 \geqslant n_2 \geqslant \cdots \geqslant n_t \geqslant \cdots \geqslant n_r \geqslant 1$.

We first consider the case $k = 3$. Since $r = d(S) \geqslant t$ and $h(S) \leqslant k - 1 = 2$, it follows that

$$0 \leqslant 2r - |S| = 2r - (p + 3 - t) \leqslant 3r - (p + 3),$$

which implies $r \geqslant \lceil p/3 \rceil + 1$. Let $A = \{2u_1, 2u_2, \ldots, 2u_r\}$, and let $B = \{u_i + u_j \mid 1 \leqslant i \neq j \leqslant r\}$. By the Dias da Silva–Hamidoune Theorem, $|B| \geqslant \min(p, 2r - 3)$. Hence

$$|A| + |B| \geqslant r + \min(p, 2r - 3) > p,$$

where the last inequality holds since $3r - 3 \geqslant 3\lceil p/3 \rceil > p$ by the assumption that p is a prime and $p \geqslant 5$. It follows that there exist $i, j, l \in \{1, 2, \ldots, r\}$ with $i \neq j$ such that $u_i + u_j = 2u_l$. Since $u_i \neq u_j$, the three elements u_i, u_j, u_l are pairwise distinct. This shows that S contains a 3-barycentric subsequence.

We now suppose $k \geqslant 4$. Then it can be easily seen that

$$t \leqslant \frac{p-2}{k} + 2 \leqslant \frac{p-2}{4} + 2 < \frac{p+1}{2} \tag{1}$$

since $p \geqslant 5$. We consider two cases for $d(S)$.

Case 1: $d(S) \geqslant t + 1$. We first claim that $n_{t+2} \leqslant k - 2$ (if $d(S) = t + 1$, then we mean $n_{t+2} = 0$). Suppose, to the contrary, that $n_1 = n_2 = \cdots = n_{t+2} = k - 1$. Then

$$\begin{aligned} |S| - (n_1 + n_2 + \cdots + n_{t+2}) &= (p + k - t) - (t+2)(k-1) = p + 2 - k(t+1) \\ &\leqslant p + 2 - k\left(\frac{p-2}{k} + 2\right) = 4 - 2k < 0, \end{aligned}$$

a contradiction, and our claim follows.

For each $i = 1, 2, \ldots, t+1$, take out one element u_i from S. Denote the remaining sequence by S'. Then $|S'| = |S| - (t+1) = p + k - 2t - 1$, and $h(S') \leqslant k - 2$ since $h(S) \leqslant k - 1$ and $n_{t+2} \leqslant k - 2$. It is clear that $|S'| \geqslant k - 2$ since

$$|S'| - (k-2) = (p + k - 2t - 1) - (k-2) = p + 1 - 2t \geqslant 0,$$

where the last inequality holds by (1). Hence, by Lemma 2.4, there exists a $(k-2)$-setpartition of S', say $B_1, B_2, \ldots, B_{k-2}$.

Let $B = \{u_i + u_j \mid 1 \leqslant i \neq j \leqslant t+1\}$. By the Dias da Silva–Hamidoune Theorem,

$$|B| \geqslant \min(p, 2(t+1) - 3) = \min(p, 2t-1) = 2t - 1,$$

where the last equality holds by (1). Let $B_1' = \{(1-k)x \mid x \in B_1\}$. Then $|B_1'| = |B_1|$, and by the Cauchy–Davenport Theorem,

$$\begin{aligned}|B_1' + B_2 + \cdots + B_{k-2} + B| &\geqslant \min(p, |S'| + |B| - (k-2)) \\ &\geqslant \min(p, (p + k - 2t - 1) + (2t - 1) - (k-2)) = p.\end{aligned}$$

It follows that $B_1' + B_2 + \cdots + B_{k-2} + B = \mathbb{Z}_p$, which implies that S contains a k-barycentric subsequence.

Case 2: $d(S) = t$. Then $S = [u_1]^{n_1}[u_2]^{n_2} \cdots [u_t]^{n_t}$. We claim that $t \geqslant 3$. Indeed, since $h(S) \leqslant k - 1$, it follows that $t(k-1) \geqslant |S| = p + k - t$, which implies $t \geqslant (p+k)/k > 2$ since $k \leqslant p - 1$. Hence $t \geqslant 3$, and our claim follows.

For each $i = 1, 2, \ldots, t$, take out one element u_i from S. Denote the remaining sequence by S'. Then $|S'| = |S| - t = p + k - 2t$, and $h(S') \leqslant k - 2$ since $h(S) \leqslant k - 1$. It is clear that $|S'| \geqslant k - 2$ since

$$|S'| - (k-2) = (p + k - 2t) - (k-2) = p + 2 - 2t \geqslant 0,$$

where the last inequality holds by (1). Hence, by Lemma 2.4, there exists a $(k-2)$-setpartition of S', say $B_1, B_2, \ldots, B_{k-2}$, such that $\||B_i| - |B_j|\| \leqslant 1$ for $1 \leqslant i \leqslant k-2$ and $1 \leqslant j \leqslant k-2$. Without loss of generality, we may assume $|B_1| \geqslant |B_2| \geqslant \cdots \geqslant |B_{k-2}| \geqslant 1$.

We claim that $|B_2| \geqslant 2$. Indeed, if $|B_2| = 1$, then we must have $|B_1| \leqslant 2$. Hence $p + k - 2t = |S'| \leqslant 2 + (k-3) = k - 1$, which implies $t \geqslant (p+1)/2$, a contradiction to (1). Thus $|B_2| \geqslant 2$, and our claim follows.

Let $B = \{u_i + u_j \mid 1 \leqslant i \neq j \leqslant t\}$. By the Dias da Silva–Hamidoune Theorem,

$$|B| \geqslant \min(p, 2t - 3) = 2t - 3,$$

where the last equality holds by (1). Notice that $|B| \geqslant 3$ since $t \geqslant 3$. We consider two cases for B_1.

Subcase 2a: B_1 and B are arithmetic progressions with the same common difference. Let $B_1' = \{(1-k)x \mid x \in B_1\}$. Since $1 - k \not\equiv \pm 1 (\text{mod } p)$, it follows by Lemma 2.5 that either B_1' and B are not arithmetic progressions with the same common difference, or that $\max(|B_1'|, |B|) \geqslant p - 1$. If $|B_1' + B| \geqslant p - 1$, then by the Cauchy–Davenport Theorem,

$$|B_1' + B_2 + \cdots + B_{k-2} + B| \geqslant |B_1' + B_2 + B| \geqslant \min(p, |B_1' + B| + |B_2| - 1) = p$$

since $|B_2| \geqslant 2$. If $|B_1' + B| < p - 1$, then, by Vosper's Theorem, we have

$$|B_1' + B| \geqslant |B_1'| + |B| = |B_1| + |B|.$$

Hence by the Cauchy–Davenport Theorem,

$$\begin{aligned}|B_1' + B_2 + \cdots + B_{k-2} + B| &\geqslant \min(p, |B_1' + B| + |B_2| + \cdots + |B_{k-2}| - (k-3)) \\ &\geqslant \min(p, |S'| + |B| - (k-3)) \geqslant \min(p, (p + k - 2t) + (2t - 3) - (k-3)) = p.\end{aligned}$$

It follows that $B_1' + B_2 + \cdots + B_{k-2} + B = \mathbb{Z}_p$, which implies that S contains a k-barycentric subsequence.

Subcase 2b: B_1 and B are not arithmetic progressions with the same common difference. Let $B_2' = \{(1-k)x \mid x \in B_2\}$. If $|B_1 + B| \geqslant p - 1$, then by the Cauchy–Davenport Theorem,

$$|B_1 + B_2' + \cdots + B_{k-2} + B| \geqslant |B_1 + B_2' + B| \geqslant \min(p, |B_1 + B| + |B_2'| - 1) = p$$

since $|B_2'| = |B_2| \geqslant 2$. If $|B_1 + B| < p - 1$, then, by Vosper's Theorem, we have

$$|B_1 + B| \geqslant |B_1| + |B|.$$

Hence by the Cauchy–Davenport Theorem,

$$\begin{aligned}|B_1 + B_2' + \cdots + B_{k-2} + B| &\geqslant \min(p, |B_1 + B| + |B_2'| + \cdots + |B_{k-2}| - (k-3)) \\ &\geqslant \min(p, |S'| + |B| - (k-3)) \geqslant \min(p, (p + k - 2t) + (2t - 3) - (k-3)) = p.\end{aligned}$$

It follows that $B_1 + B_2' + \cdots + B_{k-2} + B = \mathbb{Z}_p$, which implies that S contains a k-barycentric subsequence.

The proof of the theorem is complete. $\square$

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